Unusual Identities for Special Functions from Waveguide Propagation Analyses

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Abstract —The analysis of electromagentic wave propagation in "cylindrical" waveguides with step discontinuities leads naturally to sets of unusual identities for various special functions. In this paper we concentrate on those expressions associated with classical rectangular and circular cross-sectional geometry. From a mathematical point of view it turns out, as expected, that the identities are related to bilinear expansions for Green's functions affiliated with familiar Sturm–Liouville boundary-value problems.

I. Introduction

The three-dimensional scalar wave equation is separable in 11 distinct coordinate systems [7]. Several of these, including rectangular, cylindrical, elliptic, and parabolic, are of the form (ζ, η, z), thereby allowing the ready analysis of electromagnetic propagation in "cylindrical" waveguides having a general cross section given by $f(\zeta,\eta)=0$. If $k^2=\omega^2\mu\epsilon$ where, as usual, ω is the radian frequency of the assumed harmonic time dependence and μ and ϵ are the permeability and permittivity of the homogeneous isotropic linear medium within the waveguides, then *transverse magnetic* (TM) and *transverse electric* (TE) field solutions have the well-known form [2, pp. 291 ff.], [5, pp. 171 ff.]

TM

$$\begin{aligned}\vec{E}_t^m &= \pm i\gamma^m[\operatorname{grad}\Phi(\zeta,\eta)]\exp(\pm i\gamma^m z)\\ E_z^m &= (h^m)^2\Phi(\zeta,\eta)\exp(\pm i\gamma^m z)\\ \vec{H}_t^m &= \pm\frac{\omega\epsilon}{\gamma^m}\left[\vec{e}_z\times\vec{E}_t^m\right]\\ H_z^m &= 0\end{aligned}\tag{1}$$

with $(\gamma^m)^2=k^2-(h^m)^2$, and

TE

$$\begin{aligned}\vec{H}_t^e &= \pm i\gamma^e[\operatorname{grad}\Psi(\zeta,\eta)]\exp(\pm i\gamma^e z)\\ H_z^e &= (h^e)^2\Psi(\zeta,\eta)\exp(\pm i\gamma^e z)\\ \vec{E}_t^e &= \mp\frac{\omega\mu}{\gamma^e}\left[\vec{e}_z\times\vec{H}_t^e\right]\\ E_z^e &= 0\end{aligned}\tag{2}$$

with $(\gamma^e)^2=k^2-(h^e)^2$. In these expressions we assume that the idealized boundary condition of vanishing tangential electric field on the waveguide wall $f(\zeta,\eta)=0$ has been applied so that $\Phi(\zeta,\eta)$ and $\Psi(\zeta,\eta)$, respectively, are solutions of the following eigenvalue problems:

$$\begin{aligned}\nabla^2\Phi+(h^m)^2\Phi &= 0 \quad \text{within waveguide}\\ \Phi &= 0 \quad \text{on waveguide wall}\end{aligned}\tag{3a}$$

$$\begin{aligned}\nabla^2\Psi+(h^e)^2\Psi &= 0 \quad \text{within waveguide}\\ \frac{\partial\Psi}{\partial n} &= 0 \quad \text{on waveguide wall.}\end{aligned}\tag{3b}$$

Manuscript received May 9, 1987; revised September 28, 1987.
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IEEE Log Number 8718870.

The problems (3a) and (3b) are classical Sturm–Liouville eigenvalue problems (see [6, pp. 228 ff.], for example). If $f(\zeta,\eta)=0$ is a piecewise-smooth simple curve (a contour) enclosing a bounded domain, each problem has a countable infinity of *positive* eigenvalues. (We ignore the eigenvalue $(h^e)^2=0$ since the corresponding eigenfunction is $\Psi=$ constant, which implies $\vec{E}\equiv\vec{H}\equiv 0$.) Indeed, when the eigenvalues associated with each problem are arranged in order of increasing size, that is,

$$0<(h_1^e)^2\leqslant(h_2^e)^2\leqslant(h_3^e)^2\leqslant\cdots$$

and

$$0<(h_1^m)^2\leqslant(h_2^m)^2\leqslant(h_3^m)^2\leqslant\cdots$$

it follows that

$$(h_p^e)^2\leqslant(h_p^m)^2$$

for each $p=1,2,\cdots$ and $\lim_{p\to\infty}(h_p^e)^2=\lim_{p\to\infty}(h_p^m)^2=\infty$.

In our electromagnetic setting, the individual TM and TE waves (modes) generated by the eigenfunctions of (3a) and (3b) are independent, and their superposition leads to the most general solution of Maxwell's equations within the waveguide [2, p. 300]. Since

$$(\gamma_p^{m,e})^2=k^2-(h_p^{m,e})^2$$

for a given frequency ω only a finite number of modes of each type are propagating. Whether propagating or evanescent, however, the transverse components of the modal fields satisfy the familiar orthogonality relations

$$\int_A\vec{E}_{tp}\cdot\vec{E}_{tq}^*\,dA=0\tag{4a}$$

$$\int_A\vec{H}_{tp}\cdot\vec{H}_{tq}^*\,dA=0\tag{4b}$$

$$\int_A\vec{e}_z\cdot\left(\vec{E}_{tp}^m\times\vec{H}_{tq}^{e*}\right)dA=0.\tag{4c}$$

In these expressions, $\vec{F}^*$ designates the complex conjugate of $\vec{F}$ and the integration is performed over the entire two-dimensional waveguide cross section A. If the waves are of the *same* type, then $p\neq q$ in (4a) and (4b). When $p=q$, the customary normalizations are [2, p. 300]

$$\int_A|\vec{E}_{tp}^m|^2\,dA=|\gamma_p^m|^2\tag{5}$$

and

$$\int_A|\vec{E}_{tp}^e|^2\,dA=\omega^2\mu^2.$$

If two waveguides of cross sections A_1 and A_2, with $A_1\subset A_2$, are joined in the plane $z=$ constant, then, at the juncture, waves in the first guide excite waves in the second guide (and conversely). Matching transverse fields in the "aperture" A_1, we have, using some obvious notation,

$$\begin{aligned}{}^1\vec{E}_{tp}^e &= \Sigma_q\left(a_{pq}{}^2\vec{E}_{tq}^m+b_{pq}{}^2\vec{E}_{tq}^e\right)\\ {}^1\vec{E}_{tp}^m &= \Sigma_q\left(c_{pq}{}^2\vec{E}_{tq}^m+d_{pq}{}^2\vec{E}_{tq}^e\right).\end{aligned}\tag{6}$$

Here the coefficients a,b,c,d represent inner products between individual modes in the two waveguides. For example,

$$d_{pq}=\frac{1}{\omega^2\mu^2}\int_{A_1}{}^1\vec{E}_{tp}^m\cdot{}^2\vec{E}_{tq}^{e*}\,dA.$$

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Owing to the orthogonal character of the modes, and their normalizations, the representations (6) lead quite naturally to the following set of identities:

$$\sum_n \left[a_{pn} a_{qn}^* |^2\gamma_n^m|^2 + \omega^2\mu^2 b_{pn} b_{qn}^* \right] = \omega^2\mu^2\delta_{pq}$$

$$\sum_n \left[a_{pn} c_{qn}^* |^2\gamma_n^m|^2 + \omega^2\mu^2 b_{pn} d_{qn}^* \right] = 0$$

$$\sum_n \left[c_{pn} c_{qn}^* |^2\gamma_n^m|^2 + \omega^2\mu^2 d_{pn} d_{qn}^* \right] = |^1\gamma_p^m|^2\delta_{pq}. \qquad (7)$$

There is a related set of identities which arises from expressing modes in the second guide in terms of modes in the first guide. We begin with the analogue of (6) in which the roles of the guides are reversed. This representation is uniformly valid at least over the "aperture" A_1 and, in view of the relations (4) and (5), gives rise to

$$\sum_n \left[\frac{a_{np}a_{nq}^*}{\omega^2\mu^2} + \frac{c_{np}c_{nq}^*}{|^1\gamma_n^m|^2} \right] |^2\gamma_p^{m\,2}\gamma_q^m|^2 + \int_{A_2 - A_1} {}^2\vec{E}_{tp}^m \cdot {}^2\vec{E}_{tq}^{m*} dA = |^2\gamma_p^m|^2\delta_{pq}$$

$$\sum_n \left[\frac{a_{np}b_{nq}^*}{\omega^2\mu^2} + \frac{c_{np}d_{nq}^*}{|^1\gamma_n^m|^2} \right] |\omega\mu^2\gamma_p^m|^2 + \int_{A_2 - A_1} {}^2\vec{E}_{tp}^e \cdot {}^2\vec{E}_{tq}^{m*} dA = 0$$

$$\sum_n \left[\frac{b_{np}b_{nq}^*}{\omega^2\mu^2} + \frac{d_{np}d_{nq}^*}{|^1\gamma_n^m|^2} \right] \omega^4\mu^4 + \int_{A_2 - A_1} {}^2\vec{E}_{tp}^e \cdot {}^2\vec{E}_{tq}^{e*} dA = \omega^2\mu^2\delta_{pq}. \qquad (8)$$

These latter expressions have a slightly different character than the relations (7) since $A_1 \subset A_2$ and the modes in the first guide do not generally constitute a basis for the modes in the second guide over the whole of A_2. In both (7) and (8), however, δ_{pq} represents the familiar Kronecker delta.

II. Rectangular and Circular Geometries

Each choice of "cylindrical" system (ζ, η, z) engenders particular eigensolutions of (3a) and (3b) and hence particular TM and TE modes (1), (2). In turn, these modes give rise to the respective identities (7), (8), particularized to the special functions appropriate for the problem at hand. In this section we concentrate on the cases of rectangular (x, y) and circular (r, θ) cross-sectional geometry.

Case 1)

$$A_1: -a \leqslant x \leqslant a, -c \leqslant y \leqslant c$$

$$A_2: -b \leqslant x \leqslant b, -c \leqslant y \leqslant c \qquad (b \geqslant a).$$

The relevant identities are, after some manipulation,

$$\sum_{k=1}^{\infty} \frac{\sin^2(k\pi\alpha)}{k^2\alpha^2 - n^2} = \frac{(1-\alpha)\pi^2}{2\alpha}\delta_{n0} \qquad (9a)$$

$$\sum_{k=1}^{\infty} \frac{\sin^2(k\pi\alpha)}{[k^2\alpha^2 - n^2]^2} = \frac{\pi^2}{4n^2\alpha} \qquad (n \neq 0) \quad (9b)$$

$$\sum_{k=1}^{\infty} \frac{\cos^2(k-1/2)\pi\alpha}{(k-1/2)^2\alpha^2 - (n-1/2)^2} = 0 \qquad (9c)$$

$$\sum_{k=1}^{\infty} \frac{\cos^2(k-1/2)\pi\alpha}{[(k-1/2)^2\alpha^2 - (n-1/2)^2]^2} = \frac{\pi^2}{4(n-1/2)^2\alpha} \qquad (9d)$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - n^2\alpha^2} = \frac{1}{2n^2\alpha^2} - \frac{\pi}{2n\alpha}\cot(n\pi\alpha) \qquad (10a)$$

$$\sum_{k=1}^{\infty} \frac{1}{[k^2 - n^2\alpha^2]^2} = -\frac{1}{2n^4\alpha^4} + \frac{\pi}{4n^3\alpha^3}\cot(n\pi\alpha) + \frac{\pi^2}{4n^2\alpha^2}\csc^2(n\pi\alpha) \qquad (10b)$$

$$\sum_{k=1}^{\infty} \frac{1}{(k-1/2)^2 - (n-1/2)^2\alpha^2} = \frac{\pi}{2(n-1/2)\alpha}\tan(n-1/2)\pi\alpha \qquad (10c)$$

$$\sum_{k=1}^{\infty} \frac{1}{[(k-1/2)^2 - (n-1/2)^2\alpha^2]^2} = \frac{\pi^2}{4(n-1/2)^2\alpha^2}\sec^2(n-1/2)\pi\alpha - \frac{\pi}{4(n-1/2)^3\alpha^3}\tan(n-1/2)\pi\alpha. \qquad (10d)$$

In the above relations n is a nonnegative integer, α is a parameter with $0 < \alpha \leqslant 1$, and δ is the familiar Kronecker delta. In terms of the physical variables, $\alpha = a/b$.

Case 2)

$$A_1: 0 \leqslant r \leqslant r_1, 0 \leqslant \theta \leqslant 2\pi$$

$$A_2: 0 \leqslant r \leqslant r_2, 0 \leqslant \theta \leqslant 2\pi \qquad (r_2 \geqslant r_1).$$

Here the underlying identities turn out to be

$$(x_l')^2 \sum_{k=1}^{\infty} \frac{(x_k')^2}{[(x_l')^2 - (\alpha x_k')^2][(x_k')^2 - n^2]} \left[\frac{J_n'(\alpha x_k')}{J_n(x_k')}\right]^2 + \frac{n^2}{\alpha^2}\sum_{k=1}^{\infty} \frac{1}{(x_k)^2}\left[\frac{J_n(\alpha x_k)}{J_n'(x_k)}\right]^2 = 0 \qquad (11a)$$

$$(x_l')^2 \sum_{k=1}^{\infty} \frac{(x_k')^4}{[(x_l')^2 - (\alpha x_k')^2]^2[(x_k')^2 - n^2]} \left[\frac{J_n'(\alpha x_k')}{J_n(x_k')}\right]^2 = \frac{(x_l')^2 - n^2}{4\alpha^4} \qquad (11b)$$

$$\sum_{k=1}^{\infty} \frac{1}{[(x_l)^2 - (\alpha x_k)^2]}\left[\frac{J_n(\alpha x_k)}{J_n'(x_k)}\right]^2 = 0 \qquad (11c)$$

$$(x_l)^2 \sum_{k=1}^{\infty} \frac{1}{[(x_l)^2 - (\alpha x_k)^2]^2}\left[\frac{J_n(\alpha x_k)}{J_n'(x_k)}\right]^2 = \frac{1}{4\alpha^2} \qquad (11d)$$

and

$$\sum_{k=1}^{\infty} \frac{(x_k')^2}{\left[(x_k')^2-(\alpha x_l')^2\right]\left[(x_k')^2-n^2\right]} = \frac{1}{2\alpha x_l'} \frac{J_n(\alpha x_l')}{J_n'(\alpha x_l')} \quad \text{(12a)}$$

$$(\alpha x_l')^2 \sum_{k=1}^{\infty} \frac{(x_k')^2}{\left[(x_k')^2-(\alpha x_l')^2\right]^2\left[(x_k')^2-n^2\right]} = \frac{1}{4} + \frac{(\alpha x_l')^2-n^2}{4(\alpha x_l')^2}\left[\frac{J_n(\alpha x_l')}{J_n'(\alpha x_l')}\right]^2 \quad \text{(12b)}$$

$$(\alpha x_l)^2 \sum_{k=1}^{\infty} \frac{1}{(x_k)^2-(\alpha x_l)^2} - \sum_{k=1}^{\infty} \frac{n^2}{(x_k')^2-n^2} = -\frac{\alpha x_l}{2}\frac{J_n'(\alpha x_l)}{J_n(\alpha x_l)} \quad \text{(12c)}$$

$$(\alpha x_l)^2 \sum_{k=1}^{\infty} \frac{(x_k)^2}{\left[(x_k)^2-(\alpha x_l)^2\right]^2} = \frac{(\alpha x_l)^2-n^2}{4} + \frac{(\alpha x_l)^2}{4}\left[\frac{J_n'(\alpha x_l)}{J_n(\alpha x_l)}\right]^2. \quad \text{(12d)}$$

In these sets of relations n is again a nonnegative integer, α is a parameter with $0 < \alpha \leqslant 1$, $J_n(x)$ is the Bessel function of the first kind of order n and argument x, $J_n'(x)$ is the derivative of $J_n(x)$, and the x_k, x_k' are the nontrivial (positive) zeros of $J_n(x)$ and $J_n'(x)$, respectively, naturally ordered. In terms of the physical variables, $\alpha = r_1/r_2$.

III. Analysis of the Results

The various identities are not all independent. For example, (10b), (10d), (12b), and (12d) follow from (10a), (10c), (12a), and (12c), respectively, by differentiation with respect to α. Moreover, (9) and (10) are consequences of (11c), (11d) and (12c), (12d) in the special cases $n = \pm 1/2$.

Although perhaps unfamiliar, the identities of the previous section do not actually represent new results. Equations (9a) and (9b) are special cases of the more general identity

$$\frac{\sin \alpha z \sin(1-\beta) z}{z \sin z} = 2\sum_{k=1}^{\infty} \frac{\sin(k\pi\alpha)\sin(k\pi\beta)}{k^2\pi^2 - z^2} \quad (13)$$

with $0 \leqslant \alpha \leqslant \beta \leqslant 1$. Equations (9c) and (9d) similarly result from

$$\frac{\cos \alpha z \sin(1-\beta) z}{z \cos z} = 2\sum_{k=1}^{\infty} \frac{\cos(k-1/2)\pi\alpha \cos(k-1/2)\pi\beta}{(k-1/2)^2\pi^2 - z^2} \quad (14)$$

again with $0 \leqslant \alpha \leqslant \beta \leqslant 1$. The identities in (10) are all contained in the infinite product expansions

$$\sin z = z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right) \quad (15)$$

and

$$\cos z = \prod_{k=1}^{\infty}\left(1-\frac{z^2}{(n-1/2)^2\pi^2}\right) \quad (16)$$

(see [1, p. 75], for example).

In analogous fashion, underlying (11) are the more general identities

$$\frac{\pi}{2}\frac{J_\nu(\alpha z)}{J_\nu(z)}\{J_\nu(\beta z)Y_\nu(z) - J_\nu(z)Y_\nu(\beta z)\} = 2\sum_{k=1}^{\infty}\frac{1}{(x_k)^2-z^2}\frac{J_\nu(\alpha x_k)J_\nu(\beta x_k)}{[J_\nu'(x_k)]^2} \quad (\nu > -1) \quad (17)$$

and

$$\frac{\pi}{2}z^2\frac{J_\nu'(\alpha z)}{J_\nu'(z)}\{J_\nu'(\beta z)Y_\nu'(z) - J_\nu'(z)Y_\nu'(\beta z)\} = 2\sum_{k=1}^{\infty}\frac{(x_k')^4}{\left[(x_k')^2-z^2\right]\left[(x_k')^2-\nu^2\right]}\frac{J_\nu'(\alpha x_k')J_\nu'(\beta x_k')}{[J_\nu(x_k')]^2} \quad (\nu > -1) \quad (18)$$

both with $0 < \alpha \leqslant \beta \leqslant 1$ (see [3], [8, p. 104], and [11, p. 499]). The infinite product expansions

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\prod_{k=1}^{\infty}\left[1-\frac{z^2}{(x_k)^2}\right] \quad (\nu > -1) \quad (19)$$

$$J_\nu'(z) = \frac{(z/2)^{\nu-1}}{2\Gamma(\nu)}\prod_{k=1}^{\infty}\left[1-\frac{z^2}{(x_k')^2}\right] \quad (\nu > 0) \quad (20)$$

incorporate the identities (12) (see [1, p. 370]). In (17)–(20) the x_k, x_k' are the nontrivial (positive) zeros of the Bessel function $J_\nu(x)$, and its derivative $J_\nu'(x)$, respectively, naturally ordered.

The expressions (13), (14), (17), and (18) are important in their own right. On the left-hand side of each identity appears a Green's function (or its derivative) associated with a particular classical Sturm–Liouville problem. The right-hand sides are merely bilinear (Fourier) expansions of these Green's functions (or their derivatives) in terms of the eigenvalues and eigenfunctions of the underlying homogeneous boundary-value problems ([4, p. 292], [9, pp. 213ff.], [10, p. 415].) The relevant problems are, respectively,

(i) $\dfrac{d^2y}{dx^2} + z^2y = 0 \qquad 0 \leqslant x \leqslant 1$
$y(0) = 0$
$y(1) = 0$

(ii) $\dfrac{d^2y}{dx^2} + z^2y = 0 \qquad 0 \leqslant x \leqslant 1$
$\dfrac{dy(0)}{dx} = 0$
$y(1) = 0$

(iii) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (z^2x^2 - \nu^2)y = 0 \qquad 0 \leqslant x \leqslant 1$
$y(0)$ bounded
$y(1) = 0$

(iv) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (z^2x^2 - \nu^2)y = 0 \qquad 0 \leqslant x \leqslant 1.$
$y'(0)$ bounded
$y'(1) = 0$

IV. Summary

The analysis of electromagnetic wave propagation in "cylindrical" waveguides with step discontinuities leads naturally to sets of unusual identities for various special functions. In this paper we have discussed the expressions induced by classical rectangular and circular geometry and examined their relationship to bilinear expansions for Green's functions attendant to familiar Sturm–Liouville boundary value problems. Other coordinate systems and cross sections give rise to identities involving Mathieu functions, confluent hypergeometric functions, and so on. These topics are the subject of further investigation.

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Self-Consistent Finite/Infinite Element Scheme for Unbounded Guided Wave Problems

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Abstract —**An efficient finite-element approach for the eigenmode analysis of unbounded guided wave problems is described using decay-type infinite elements. To determine an optimum set of decay parameters, two algorithms based on successive approximation are presented and their validity is checked via the application to an optical fiber problem.**

I. INTRODUCTION

It is well recognized that difficulty is frequently encountered when one wants to solve unbounded field problems using finite elements. To overcome this difficulty, these unbounded domains have in the past been dealt with in various ways, all of which have strengths and weaknesses. To date the main methods in guided wave problems have been simple truncation [1]–[4], the use of analytical far-field solutions [5], the decay-type infinite element approach [6], [7], the exterior finite element approach [8], and the conformal mapping technique [9]. The simplest technique among them is undoubtedly the simple truncation, in which the unbounded domain is truncated to a finite size. However, this technique involves a very large number of nodal points when the field extends farther away from the guiding region. Among other methods the decay-type infinite element approach, in which a finite element is extended to infinity, is often simple and economical and has now been applied successfully to a wide range of problems [10], [11]. A pending question in applying this method is the determination of unknown parameters involved which represent decaying behavior in a far-field region. Although almost all of the authors to date have mentioned this problem, no systematical algorithm for determining the decay parameters has yet been developed [6], [7], [10], [11].

Manuscript received May 20, 1987; revised October 5, 1987
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IEEE Log Number 8718868.

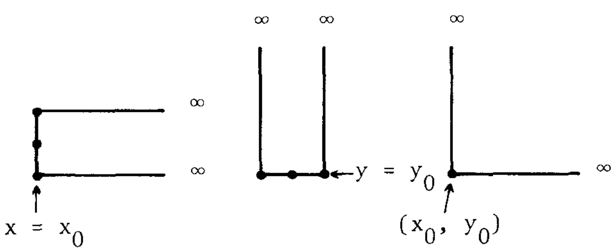

Fig. 1. Infinite elements.

In this paper, a self-consistent finite/infinite element scheme that can be used for the eigenmode analysis of unbounded dielectric waveguide problems is developed. To determine the decay parameters involved, two algorithms based on successive approximation are proposed and their validity is examined by means of the application to an optical fiber problem. By using these algorithms, an optimum set of decay parameters is readily obtainable in a self-consistent iterative way.

II. DETERMINATION OF AN OPTIMUM SET OF DECAY PARAMETERS

Consider strip-like infinite elements shown in Fig. 1 and expand the field ϕ in each element as

$$\phi = \{N\}^T\{\phi\}_e \qquad (T\text{: transposition}) \tag{1}$$

where $\{N\}$ is the shape function vector of the infinite elements and $\{\phi\}_e$ is the nodal vector for each element.

As a trial function for semi-infinite directions, we choose the following decay function:

$$f(\xi; c) = \exp\{-c(\xi-\xi_0)^p\} \qquad (c>0, p>0.5) \tag{2}$$

where c is the unknown decay parameter and $(\xi, \xi_0, c) = (x, x_0, \alpha_x), (y, y_0, \alpha_y)$. If p is set to unity, (2) is reduced to the exponential function [6], [7], [10], [11]; we choose $p=1$ in the following description.

To determine systematically the best value of c, we propose here the following two algorithms:

A. A Method Utilizing the Field Profile in a Finite Element Region

Fig. 2 shows a schematic illustration of a field profile on the axes. We approximate the field ϕ near the points x_0, y_0 as

$$\phi(x,0) = u_0 \exp\{-\alpha_x(x-x_0)\} \tag{3}$$

$$\phi(0,y) = v_0 \exp\{-\alpha_y(y-y_0)\}. \tag{4}$$

If we choose other points (x_1, u_1) and (y_1, v_1) corresponding to the nodes in a finite element region, the unknown parameters